Def

A function f is called differentiable at c if

$f'(c) := \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists. $f'(c)$ is called the derivative of f at c

1. Show that $f(x) = |x|$, $x \in \mathbb{R}$ is not differentiable at 0.

Pf: Note that $f(x) = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$

Then $$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

Thus $$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = 1$$

and $$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = -1$$

Therefore $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exist.

Hence f is not differentiable at 0.

2. Show that $f(x) = x^{\frac{1}{3}}$, $x \in \mathbb{R}$ is not differentiable at 0.

Pf: Note that $\dfrac{f(x) - f(0)}{x - 0} = \dfrac{f(x)}{x} = \dfrac{x^{\frac{1}{3}}}{x} = x^{-\frac{2}{3}}$.

Suppose $\lim\limits_{x \to 0} x^{-\frac{2}{3}} = L$ exists.

Take $x_n = \frac{1}{n}$. Then $x_n \to 0$ as $n \to \infty$

By Sequential Criterion,

$$x_n^{-\frac{2}{3}} = n^{\frac{2}{3}} \to L \quad \text{as} \quad n \to \infty.$$

But $n^{\frac{2}{3}} \to \infty$ as $n \to \infty$ by AP.

Contradiction!

3(a) Show that $f(x) = \begin{cases} x^2, & x \text{ rational}, \\ 0, & x \text{ irrational} \end{cases}$ is differentiable at 0 and $f'(0) = 0$.

Pf: Note that $\dfrac{f(x) - f(0)}{x - 0} = \dfrac{f(x)}{x} = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational}. \end{cases}$

For any $\varepsilon > 0$, if $|x| < \varepsilon$,

$\left| \dfrac{f(x) - f(0)}{x - 0} \right| < \varepsilon$ in both rational and irrational cases.

Thus $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0$.

Hence $f'(0) = 0$

(b) What about $f(x) = \begin{cases} x, & x \text{ rational}, \\ 0, & x \text{ irrational} \end{cases}$?

Claim: $f'(0) := \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exist.

Pf: Suppose not.

Write $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = L$.

Note that $\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = \begin{cases} 1, & x \text{ rational}, \\ 0, & x \text{ irrational}. \end{cases}$

By density of $\mathbb{Q}$, there exists a sequence $(x_n) \in \mathbb{Q}$ s.t. $x_n \to 0$ as $n \to \infty$.

Then $\frac{f(x_n) - f(0)}{x_n - 0} \to 1$ as $n \to \infty$.

By density of $\mathbb{R} \setminus \mathbb{Q}$, there exists a sequence $(y_n) \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $y_n \to 0$ as $n \to \infty$.

Then $\frac{f(x_n) - f(0)}{x_n - 0} \to 0$ as $n \to \infty$.

But by Sequential Criterion,

$\frac{f(x_n) - f(0)}{x_n - 0} \to L$ as $n \to \infty$ and

$\frac{f(y_n) - f(0)}{y_n - 0} \to L$ as $n \to \infty$.

Then $1 = L = 0$.

Contradiction!

4(a) Show that $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$ is differentiable at 0 and $f'(0) = 0$.

Pf: Note that $\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = \frac{x^2 \sin\frac{1}{x}}{x} = x \sin\frac{1}{x}$

and $-1 \leq \sin\frac{1}{x} \leq 1$.

Then $-x \leq \frac{f(x) - f(0)}{x - 0} \leq x$.

Since $\lim_{x \to 0} x = \lim_{x \to 0} (-x) = 0$, by Squeeze Theorem,

$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0$.

Hence, $f'(0) = 0$.

(b) What about $f(x) = \begin{cases} x \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0 \end{cases}$?

Claim: $f'(0) := \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exist.

Pf: Suppose not.

Write $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = L$.

Note that $\frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x} = \frac{x\sin\frac{1}{x}}{x} = \sin\frac{1}{x}$.

Take $x_n = \frac{1}{2\pi n}$.

Then $\frac{f(x_n)-f(0)}{x_n-0} = \sin 2\pi n = 0 \to 0$ as $n\to\infty$

Take $y_n = \frac{1}{(2n+\frac{1}{2})\pi}$.

Then $\frac{f(y_n)-f(0)}{y_n-0} = \sin(2n+\frac{1}{2})\pi = 1 \to 1$ as $n\to\infty$.

Since $x_n\to 0$ and $y_n\to 0$ as $n\to\infty$,

by Sequential Criterion,

$$\frac{f(x_n)-f(0)}{x_n-0} \to L \quad \text{and} \quad \frac{f(y_n)-f(0)}{y_n-0} \to L$$

as $n\to\infty$.

Therefore $0 = L = 1$.

Contradiction!

5(a) Suppose $f'(c) := \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists.

Show that $f'(c) = \lim_{n\to\infty} n\left[f(c+\frac{1}{n}) - f(c)\right]$.

Pf: Take $x_n = c+\frac{1}{n}$. Then $x_n\to c$ as $n\to\infty$.

Since $f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$, by Sequential Criterion,

$$f'(c) = \lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} = \lim_{n\to\infty} \frac{f(c+\frac{1}{n}) - f(c)}{c+\frac{1}{n} - c}$$

$$= \lim_{n\to\infty} n\left[f(c+\tfrac{1}{n}) - f(c)\right]$$

(b) Give an example to show the existence of $\lim_{n\to\infty} n\left[f(c+\frac{1}{n}) - f(c)\right]$ does not imply $f'(c)$ exists.

Example 1

$f(x) = |x|$, $c = 0$.

Then $\lim_{n\to\infty} n\left[f(c+\frac{1}{n}) - f(c)\right] = \lim_{n\to\infty} n f(\frac{1}{n})$

$$= \lim_{n\to\infty} n \cdot \frac{1}{n} = 1.$$

By Q1, $f'(0)$ does not exists.

Example 2

$$f(x) = \begin{cases} x, & x \text{ rational}, \\ 0, & x \text{ irrational}. \end{cases} \qquad c = 0$$

Then $\lim_{n\to\infty} n[f(c+\frac{1}{n})-f(c)] = \lim_{n\to\infty} nf(\frac{1}{n}) = \lim_{n\to\infty} n\cdot\frac{1}{n} = 1.$

By Q2, $f'(0)$ does not exist.